ON A COMPLETE SYSTEM OF SOLUTIONS OF THE LINEARIZED NAVIER-STOKES EQUATIONS AND ITS APPLICATION TO BOUNDARY VALUE PROBLEMS INVOLVING FLOW ABOUT SPHERES

(O NEKOTOROI POLNOI SISTEME RESHENII LINEARIZOVANNYKH URAVNENII NAV'E-STOKSA I EE PRIMENENII K KRAEVYM ZADACHAM OB OBTEKANII SFER)

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In the present paper a complete system of solutions of the linearized Navier-Stokes equations in the case of the steady motion of a viscous fluid in the absence of external forces is given. The system of solutions was found by the method of separation of variables given in [1]. For the solutions obtained formulas of transformation from one center to another, or transfer formulas, are then deduced by means of which, as is done in [2], it is possible to solve various boundary value problems involving flow about spheres which reduce to solving an infinite system of linear algebraic equations. These solutions, deduced here by the method of [1], can in principle also be obtained from the general solution of the system (1.1) given by Lamb [3].

1. The system of normal solutions and the finding of transfer formulas for them. We shall consider the linearized equations of the steady motion of a viscous fluid in the absence of external forces

$$\nu\Delta \mathbf{v} = \frac{1}{\rho}\operatorname{grad} p, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

where $\mathbf{v}$ is the velocity, p the pressure, ν the kinematic viscosity coefficient, and ρ the density of the fluid. Taking the curl of the left and right sides of the first equation, we eliminate p and obtain the system of equations

$$\operatorname{rot}\Delta\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

Applying the method of separation of variables of [1] to the system (1.2), we obtain six types of solutions (we shall call them the normal solutions) in spherical coordinates: the exterior normal solutions $\mathbf{u}_{ln}$, $\mathbf{v}_{ln}$, $\mathbf{w}_{ln}$ and the interior normal solutions $\mathbf{p}_{ln}$, $\mathbf{q}_{ln}$, $\mathbf{r}_{ln}$ (the exterior normal solutions are used for the solution of boundary value problems in unbounded domains and the interior normal solutions in bounded domains).

The solutions $\mathbf{u}_{ln}$ and $\mathbf{p}_{ln}$ have the form

$$\begin{aligned} \mathbf{u}_{ln} &= r^{-l}\left(Y_{ln}\,\mathbf{e}_r - \frac{l-2}{l(l+1)}\frac{\partial}{\partial\theta}Y_{ln}\,\mathbf{e}_\theta - \frac{l-2}{l(l+1)}\frac{\mathbf{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\right) \\ \mathbf{p}_{ln} &= r^{l+1}\left(Y_{ln}\,\mathbf{e}_r + \frac{l+3}{l(l+1)}\frac{\partial}{\partial\theta}Y_{ln}\,\mathbf{e}_\theta + \frac{l+3}{l(l+1)}\frac{\mathbf{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\right) \qquad (1.3) \\ Y_{ln}(\theta,\varphi) &= P_{ln}(\cos\theta)\,e^{in\varphi} \qquad (l=0,1,2,\ldots;\ -l\leqslant n\leqslant l) \end{aligned}$$

Here P_{ln} are the associated Legendre functions which are defined by the formula

$$P_{ln}(x) = \frac{(1-x^2)^{n/2}}{2^l\, l!}\frac{d^{l+n}}{dx^{l+n}}(x^2-1)^l$$

But the solutions $\mathbf{v}_{ln}$, $\mathbf{w}_{ln}$, $\mathbf{q}_{ln}$ and $\mathbf{r}_{ln}$ coincide with similarly designated solutions of the static equation of elasticity [2]. The explanation of this is that for these solutions div $\mathbf{v} = 0$ (e.g. see [2]); consequently, being solutions of the static equation of elasticity

$$\Delta\mathbf{v} + \frac{1}{1-2\sigma}\,\mathrm{grad\,div}\,\mathbf{v} = 0$$

they satisfy the equation $\Delta\mathbf{v} = 0$ and hence the equations (1.2) also.

Transfer formulas for the solutions $\mathbf{v}_{ln}$ and $\mathbf{w}_{ln}$ are given in [2].

We shall find a transfer formula for $\mathbf{u}_{ln}$ which expresses the solution referred to a system of spherical coordinates with origin at the point O_1 (see figure) by means of interior normal solutions referred to a system of coordinates with origin at the point O_2 (both systems have a common axis z passing through the points O_1 and O_2 and parallel axes x and y).

We shall seek it in the form

$$\mathbf{u}_{ln}(r_1,\theta_1,\varphi) = \sum \alpha_{lkn}\,\mathbf{p}_{kn}(r_2,\theta_2,\varphi) + \sum \beta_{lkn}\,\mathbf{q}_{kn}(r_2,\theta_2,\varphi) + \sum \gamma_{lkn}\,\mathbf{r}_{kn}(r_2,\theta_2,\varphi) \qquad (1.4)$$

where the summation limits are simultaneously determined with the presently unknown coefficients α_{lkn}, β_{lkn}, γ_{lkn} (for convenience in subsequent calculations we shall take the vectors $\mathbf{r}_{kn}$ in this formula to be reduced by $[2l(l+1)]^{-1/2}$).

To find the coefficients we shall rely, in the same way as was done in [2], on the transfer formula for spherical functions [3, p.136]

$$\frac{Y_{ln}(\theta_1, \varphi)}{r_1^{l+1}} = \sum_{k=n}^{\infty} \frac{(-1)^{l-n}}{d^{l+k+1}} \frac{(l+k)!}{(k+n)!\,(l-n)!} Y_{kn}(\theta_2, \varphi)\, r_2^{k} \tag{1.5}$$

where $r_2 < d$ (d is the distance between the points O_1 and O_2 in the figure).

We shall take into account that we have $\Delta\mathbf{v} = 0$ for the solutions $\mathbf{r}_{kn}$ and $\mathbf{q}_{kn}$. But for the solutions $\mathbf{u}_{ln}$ and $\mathbf{p}_{kn}$ we have $\Delta\mathbf{v} = \operatorname{grad} \varphi$, where φ is a harmonic function since

$$\Delta\mathbf{v} = \operatorname{grad} \frac{P}{\nu\rho}$$

for solutions of system (1.1), and div $\Delta\mathbf{v} = 0$ by virtue of the second equation of this system.

Therefore, to find the coefficients α_{lkn} it is convenient to apply the Laplace operator to the left and right sides of the equality (1.4). We obtain

$$\Delta\mathbf{u}_{ln}(r_1, \theta_1, \varphi) = \sum \alpha_{lkn}\, \Delta\mathbf{p}_{kn}(r_2, \theta_2, \varphi)$$

or after expanding this equality into φth components

$$\operatorname{grad}_{\varphi}\left[\frac{(2l-1)Y_{ln}(\theta_1,\varphi)}{(l+1)\, r_1^{l+1}}\right] = \sum \alpha_{lkn} \operatorname{grad}_{\varphi}\left[\frac{2k+3}{k} Y_{kn}(\theta_2, \varphi)\, r_2^{k}\right]$$

By virtue of (1.5) we obtain

$$\alpha_{lkn} = \frac{(-1)^{l-n}}{d^{l+k+1}} \frac{(l+k)!}{(k+1)!\,(l-n)!} \frac{k(2l-1)}{(2k+3)(l+1)} \qquad (k = n, n+1, \ldots) \tag{1.6}$$

Later we shall use the condition that in view of (1.2)

$$\Delta(\operatorname{rot} \mathbf{v}) = 0$$

that is the Cartesian components of rot $\mathbf{v}$ are harmonic functions. But for the solutions $\mathbf{p}_{kn}$ (e.g. see [2])

$$\operatorname{rot} p_{kn} = 0$$

Therefore, to find the coefficients γ_{lkn} it is convenient to take the curl of the left and right sides of the equality (1.4) and to write out the vector equality obtained in Cartesian components, for example, in the zth component. We obtain

$$\operatorname{rot}_z \mathbf{u}_{ln}(r_1, \theta_1, \varphi) = \sum_{k=n}^{\infty} \alpha_{lkn} \operatorname{rot}_z \mathbf{p}_{kn}(r_2, \theta_2, \varphi) + \sum \gamma_{lkn} \operatorname{rot}_z \mathbf{r}_{kn}(r_2, \theta_2, \varphi), \qquad (1.7)$$

After writing out the zth component of the curls corresponding to the vectors we obtain spherical functions on the left and right sides of (1.7). Using (1.5) and (1.6), we find

$$\gamma_{lkn} = \frac{(-1)^{l-n-1}}{d^{l+k}} \frac{(l+k)!}{(k+n)!\,(l-n)!} \frac{2n\,(2l-1)}{k\,(k+1)\,l\,(l+1)} \qquad (k = n+1, n+2, \ldots) \quad (1.8)$$

To find the coefficients β_{lkn} we write out the vector equality (1.4) in φth components. In addition we shall use the recurrence formulas

$$\frac{P_{ln}(x)}{\sqrt{1-x^2}} = -\frac{(l+n-1)(l+n)\,P_{l-1,\,n-1}(x) + P_{l-1,\,n+1}(x)}{2n}$$

$$\frac{P_{ln}(x)}{\sqrt{1-x^2}} = -\frac{(l-n+1)(l-n+2)\,P_{l+1,\,n-1}(x) + P_{l+1,\,n+1}(x)}{2n}$$

which can be obtained with the help of formulas available in [4].

The obtained equality has to be fulfilled separately for associated functions with second index $n-1$ and $n+1$. For the latter functions it has the form (after reducing by $-1/2i$)

$$-\frac{l-2}{l\,(l+1)} \frac{Y_{l-1,\,n+1}}{r_1^{\,l}} = \sum_{k=n}^{\infty} \alpha_{lkn} \frac{k+3}{k\,(k+1)} Y_{k+1,\,n+1}\, r_2^{\,k+1} +$$
$$+ \sum \beta_{lkn} \frac{Y_{k-1,n+1}}{k} r_2^{\,k-1} + \sum_{k=n+1}^{\infty} \gamma_{lkn} Y_{k,\,n+1}\, r_2^{\,k}$$

Hence, using (1.5), we find

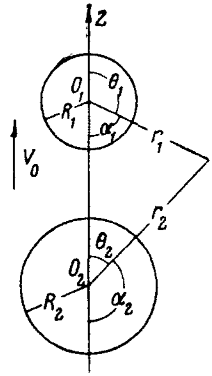
$$\beta_{lkn} = \frac{(-1)^{l-n+1}}{d^{l+k-1}} \frac{(k+l-2)!}{(k+n)!\,(l-n)!} \left[\frac{\sigma_{k-1,\,l} + 2n^2\,\tau_{k-l,\,l}}{(k-1)(2k-1)\,l\,(l+1)}\right] \qquad (1.9)$$

where

$$\sigma_{kl} = l\,[(2l-1)\,k^4 + 2l\,(l+1)\,k^3 + l\,(3l+1)\,k^2 + l\,(l+1)\,k] \qquad (1.10)$$
$$\tau_{kl} = (l-2)\,k^3 + (l^2-3l-1)\,k^2 - l\,(l+1)\,k \qquad (k = n+2,\ n+3, \ldots)$$

The transfer formulas for the exterior normal solutions are valid inside a sphere of radius d with center at the point O_1 (see figure) since formula (1.5) holds in this region. In an analogous manner transfer formulas for the exterior of this sphere as well as for the interior normal solutions can be found. For this it is necessary to use the corresponding formulas for spherical functions [3].

2. Application of the solutions and transfer formulas obtained to the solution of boundary value problems. We shall examine the flow of a viscous fluid about two spheres under the adopted assumptions that the velocity of the flow at infinity is equal to v_0 and is directed, for example, parallel to a line joining the centers of the spheres (figure).



The unknown velocity vector $\mathbf{v}$, which is a solution of equations (1.1) or (1.2), must satisfy the conditions

$$\mathbf{v} = 0 \text{ on the spheres } \mathbf{v}|_\infty = \mathbf{v}_0 \tag{2.1}$$

The uniqueness theorem holds for the formulated problem.* We shall seek the solution of the boundary value problem in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, where $\mathbf{v}_1|_\infty = 0$ by virtue of condition (2.1).

Since (see [2])

$$\mathbf{v}_0 = |\mathbf{v}_0| (\cos\theta\, \mathbf{e}_r - \sin\theta \mathbf{e}_\theta) = |\mathbf{v}_0| \mathbf{q}_{10} \tag{2.2}$$

it is then natural to seek $\mathbf{v}_1$ in the form of a series in the axisymmetric solutions $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0} (\mathbf{w}_{l0} \equiv 0)$.

We shall seek $\mathbf{v}_1$ in the form of a series in the vectors $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0}$ which are referred to the centers of both spheres (see figure)

$$\mathbf{v}_1 = \sum_{l=0}^{\infty} A_{l1} \mathbf{u}_{l0}(r_1, \theta_1) R_1^{l+1} + B_{l1} \mathbf{v}_{l0}(r_1, \theta_1) R_1^{l+3} +$$
$$+ \sum_{l=0}^{\infty} A_{l2} \mathbf{u}_{l0}(r_2, \theta_2) R_2^{l+1} + B_{l2} \mathbf{v}_{l0}(r_2, \theta_2) R_2^{l+3} \tag{2.3}$$

Hence we have four series of unknown coefficients for the determination of which there are four boundary conditions

$$v_{1r} = 0 \qquad v_{1\theta} = 0 \tag{2.4}$$

on the surface of each sphere.

To require that the conditions (2.4) be fulfilled on the surface of

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* Its proof is not presented here. The validity of this theorem follows, for example, from the uniqueness theorem for the generalized solution proved in [5].

the first sphere, we express $\mathbf{u}_{l0}(r_2, \theta_2)$ and $\mathbf{v}_{l0}(r_2, \theta_2)$ by means of transfer formulas in terms of the normal solutions which are referred to the center of the sphere, i.e. which are functions of r_1 and θ_1. In addition, it will first be necessary to use the angles α_1 and α_2 and then to take into consideration that

$$\cos\alpha_i = -\cos\theta_i \qquad (i = 1, 2), \qquad P_{ln}(-x) = (-1)^{l-n} P_{ln}(x)$$

The boundary conditions on the surface of the second sphere are satisfied by an analogous treatment.

We shall obtain an infinite system of linear algebraic equations for finding the unknown coefficients which is analogous to the system obtained in [2], but is simpler since mirror images are absent in the problem under consideration. Performing a substitution for the unknowns, as was also done in [2], we give to the system the form

$$z_k + \sum_{l=1}^{\infty} C_{kl} z_l = b_k \tag{2.5}$$

whose matrix is a completely continuous operator [2] in the Hilbert space l_2.

By virtue of the validity of the uniqueness theorem for the given problem there exists the following case of a Fredholm alternative: the system (2.5) has a unique solution, belonging to l_2, for any value of the right side (b_k vanishes for all k greater than some value because b_k always belongs to l_2). This solution can be found by the method of truncation, or reduction, and by the method of successive approximations, at least in the region of regularity (the system is regular for sufficiently separated spheres).

The proof that the constructed series in sum with $\mathbf{v}_0$ gives a solution to the formulated boundary value problem is carried out in the same manner as in [2].

The problem of the flow about two spheres when $\mathbf{v}_0$ is directed perpendicularly to the line joining their centers is solved in an analogous manner (only here it is necessary to use the solutions with the second index *1*). Thus by virtue of the linearity of the equations under consideration the problem is solved for arbitrary direction of the velocity $\mathbf{v}_0$.

In a similar way it is then possible to solve the problem of the flow of a viscous fluid about three or more spheres. In addition, besides the transfer formulas, it will be necessary to use also rotation formulas [2] which express a normal vector in the given system of coordinates in terms of the normal vectors under consideration in a system of coordinates obtained from the given system by a rotation.

Additionally, in a manner analogous to that done in [2,6], it is possible to solve the boundary value problems (with certain simplifications) of the flow of a viscous fluid about spherical cavities in a half-space and in a strip.

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